EQUILIBRIUM OF A LOOSE MEDIUM BETWEEN VERTICAL WALLS

Yu. A. Buevich

UDC 539.215

The plane and axisymmetric problems of the stress distribution in a loose medium bounded by vertical walls are studied on the basis of a model according to which the limiting condition for the friction force holds only at the walls and the medium in the interior is not in a limiting state. There is an analysis of the static equilibrium of a loose medium in a container and of the dynamic equilibrium of a densely packed bed moving under the influence of gravity or a piston.

The stress distribution which arises when loose materials are poured into bins and other containers or during the motion of granular beds is of considerable practical interest in many technological fields. For example, study of this distribution is required in calculations of the strength of bins and in working out appropriate standards; detailed information about the compressional stresses acting on the individual particles is required for designing catalytic chemical reactors using stationary granular bed, because of the limited strength of the catalyst grains; etc. Furthermore, study of the static and dynamic equilibrium of a granular bed in various types of apparatus is extremely important for analyzing the transition of the bed into a fluidized state and for carrying out calculations for the transport of loose media under dense-bed conditions.

Ordinarily, the equilibrium of loose media in bins, etc., is studied under the assumption that the system is in its limiting equilibrium state, as a particular case of the more general problem of the pressure exerted by a medium on bulkheads [1, 2]. The adoption of this assumption makes it possible to use the limiting relation between the normal and tangential stresses (as a rule, the Coulomb law) at the slipping surfaces to close the system of equilibrium equations, to find the characteristics of this system, and to then determine the stress field in the various regions into which the entire volume filled by loose medium is divided.

The numerical results obtained in this manner are difficult to interpret and are extremely inconvenient for practical purposes; accordingly, particular approximate models have been formulated from which comparatively simple analytic results can be obtained. However, models of this type are usually of limited value. For example, the model which has been worked out most thoroughly [3, 4] is based on both the assumption that the limiting state is reached at all points in the loose medium in the container and the assumption that the horizontal normal stresses are independent of the horizontal coordinate. The stress fields obtained on the basis of these assumptions in [3, 4] do not satisfy the equilibrium equations (in fact, these assumptions are mutually consistent only in the trivial case in which there is no horizontal motion at all).

Furthermore, the very hypothesis that a limiting equilibrium is established is extremely dubious for the case of a loose medium in a container. It follows from an analysis of the plane and axisymmetric problems in [5, 6], carried out in connection with the question of the beginning of fluidization of a granular bed, that the following equation holds in the limiting state:

$$\sigma_x\sigma_z = \tau_{xz}^2 = \tau^2 \tag{1}$$

Institute of Problems of Mechanics, Academy of Sciences of the USSR, Moscow. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 28, No. 3, pp. 455-464, March, 1975. Original article submitted April 3, 1974.

(for simplicity we assume ideally loose media alone; the generalization to the case of a medium with adhesion is quite straightforward, as will be shown below). Obviously, we have $\tau = 0$ at the axis or at the symmetry plane, so that one of the normal stresses must also vanish at this plane lower axis. This conclusion contradicts experimental data, at least for deep granular beds. Actually, the state of the loose medium can differ significantly from the limiting state. Examples of these "nonlimiting" states are the simple stress state corresponding to the solution of the familiar Hertz problem of the elastic deformation of a volume of spherical particles and the more common states with irreversible plastic deformation treated by Cherepanov [7].

From the physical standpoint, the appearance of regions in the nonlimiting state is quite natural. If the angle δ_w of the boundary friction at the walls is smaller than the internal-friction angle δ_i, the settling of the medium (e.g., under the influence of its weight) leads to a situation such that the limiting relation

$$|\tau_n| = \sigma_n \operatorname{tg} \delta \tag{2}$$

is reached at the walls in the case $\delta = \delta_w$ earlier than the analogous relation is reached in the volume, i.e., with $\delta = \delta_i$. Since the tangential stress is continuous near the walls, this stress agrees with (2) and at any rate is lower than its limiting value which follows from the Coulomb law. If, on the other hand, we have $\delta_w > \delta_i$, then Eq. (2) remains valid at the walls, but we have $\delta = \delta_i$. In many cases (in particular, those discussed below) it is precisely at the walls where the tangential stress reaches its maximum. Clearly, the state of the medium in the interior is not the limiting state in such cases, even if $\delta_w > \delta_i$. Physically, these arguments mean that the wall-friction force in (2) and the resulting normal stresses are completely capable of balancing the gravitational force and preventing plastic flow, which could lead to a further evolution of the stress state all the way to the limiting state.

In a study of nonlimiting states the internal-friction angle should be replaced by some other parameter to describe the loose material. Following [7], we consider the homogeneous compression of a loose medium along one of the axes (e.g., the z axis) under the condition that the medium is bounded by solid walls parallel to this axis. Clearly, the stress σ_z is homogeneous and equal to the applied (external) pressure. We assume that the resulting transverse normal stresses are approximately proportional to σ_z, and we adopt the coefficient of this proportionality, σ_z, as the basic characteristic of the loose material. If there is elastic deformation alone- the value of σ_z is governed by the shape of the particles, the nature of their packing, and the Poisson ration. More generally, this quantity is also governed by the plastic deformations which occur during the establishment of the equilibrium and by the resulting irreversible elastic deformations (it is the latter which cause the thrust forces on the walls which do not vanish when the gravitational force on the medium is neutralized, the formation of arches during free crumbling, etc.). Accordingly, $\varkappa$ depends on the history of the conversion of the medium to its equilibrium state--the method used to pack it, various dynamic effects, etc. If the irreversible plastic deformation is pronounced, $\varkappa$ cannot be much larger than one [7].

We consider a nonlimiting equilibrium of the type discussed above between plane vertical walls or within a vertical cylinder, assuming for simplicity that the problem is symmetric with respect to the plane or axis x' = 0. The coordinate system and geometric parameters of the problem used below are explained in Fig. 1. In the plane problem, x' is a horizontal Cartesian coordinate; in the axisymmetric problem it is the radial coordinate. We introduce the dimensionless quantities

$$x = x'/R, \quad z = z'/R, \quad h = H/R. \tag{3}$$

In terms of the coordinates in (3), the equilibrium equations are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial z} = 0, \quad \frac{\partial \tau}{\partial x} + \frac{k\tau}{x} + \frac{\partial \sigma_z}{\partial z} = \gamma R = \Gamma, \tag{4}$$

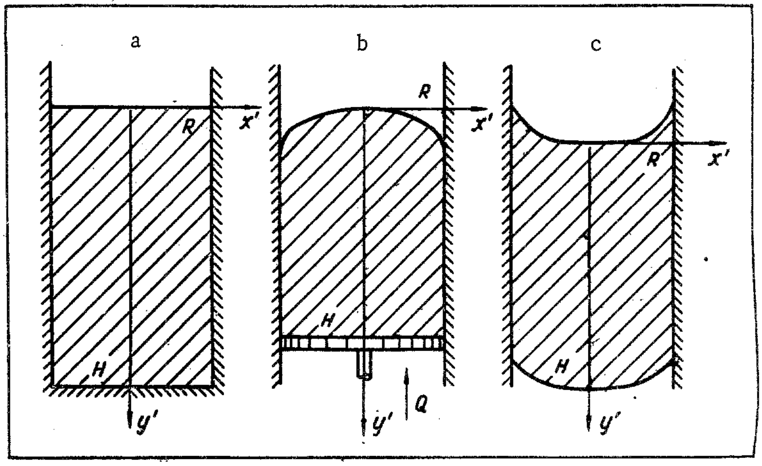

Fig. 1. Formulation of the problem. a) Equilibrium of the loose medium in the container; b) dynamic equilibrium during motion of the piston (the shape of the free surface corresponds to upward motion); c) free fall of the dense bed between the wall.

where the plane problem corresponds to k = 0 and the axisymmetric problem corresponds to k = 1. We write the boundary conditions at the walls and the plane or the symmetry axis and the symmetry conditions as

$$\tau = \pm \alpha\sigma_x \ (x = 1), \quad \sigma_x = \varkappa\sigma_z \ (x=0), \quad \alpha = \operatorname{tg}\min\{\delta_w, \delta_i\},$$
$$\sigma_x(x, z) = \sigma_x(-x, z), \quad \sigma_z(x, z) = \sigma_z(-x, z), \quad \tau(x, z) = -\tau(-x, z). \tag{5}$$

In Eqs. (4) we have used the sign convention customary in the statics of loose media [1], according to which compressional stresses are positive. We also note that the first equation in (4) is written under the usual assumption that the radial and azimuthal normal stresses are approximately equal (see, e.g., [6]). The exact equality of these quantities at the x = 0 axis or plane follows from the symmetry of the problem.

The upper and lower signs in (5) correspond to the "minimum" and "maximum" stress states. The first of these states is established when the wall-friction forces prevent settling of the loose medium and are directed upward, i.e., when the medium is exerting a force on the wall and the base of the container or during the fall of a dense bed. The second state is established during motion of a granular bed upward under the influence of a force on the base of the container, treated as a piston. In this case the frictional forces opposing the motion are directed downward.

The conditions to be imposed on the upper and lower surfaces of the bed depend on the nature of the equilibrium under study, and it is more convenient to examine these conditions separately for the concrete problems discussed below.

We seek a solution of Eqs. (4) in the form of the series

$$\sigma_x = A_0(z) + \sum_{n=1}^{\infty} A_n(z)\, x^{2n}, \quad \sigma_z = \sum_{n=1}^{\infty} B_n(z)\, x^{2(n-1)}, \quad \tau = \sum_{n=1}^{\infty} C_n(z)\, x^{2n-1}, \tag{6}$$

which satisfy the symmetry conditions in (5). From (4) and (5) we find the following problem for the coefficients A_0, A_1, B_1, C_1:

$$2A_1 + \frac{dC_1}{dz} = 0, \quad (1+k)C_1 + \frac{dB_1}{dz} = \Gamma,$$
$$C_1 = \pm\alpha(A_0 + A_1), \quad A_0 = \varkappa B_1. \tag{7}$$

Depending on the sign of the parameter

$$T = 1 - 2(1+k)\varkappa\alpha^2 \tag{8}$$

the eigenvalues of problem (7) can be either real or complex. If T > 0, the eigenvalues are real,

$$\lambda_{1,2} = \alpha^{-1}\mu_{1,2}, \quad \mu_1 = \mp 1 + \nu, \quad \mu_2 = \mp 1 - \nu, \quad \nu = |T|^{1/2} \tag{9}$$

and the solution of (7) is determined within two constants, a and b:

$$\begin{gathered}
A_0 = \pm \frac{\Gamma}{(1+k)\alpha} + \frac{\pm 1 + \nu}{2\alpha} a e^{\lambda_1 z} + \frac{\pm 1 - \nu}{2\alpha} b e^{\lambda_2 z}, \\
A_1 = -\frac{1}{2}(\lambda_1 a e^{\lambda_1 z} + \lambda_2 b e^{\lambda_2 z}), \quad C_1 = \frac{\Gamma}{1+k} + a e^{\lambda_1 z} + b e^{\lambda_2 z}, \\
B_1 = \pm \frac{\Gamma}{(1+k)\varkappa\alpha} + \frac{\pm 1 + \nu}{2\varkappa\alpha} a e^{\lambda_1 z} + \frac{\pm 1 - \nu}{2\varkappa\alpha} b e^{\lambda_2 z}.
\end{gathered} \tag{10}$$

If T < 0, the eigenvalues are complex,

$$\lambda_1 = \mp \lambda + i\omega, \quad \lambda_2 = \mp \lambda - i\omega, \quad \lambda = \alpha^{-1}, \quad \omega = \alpha^{-1}\nu \tag{11}$$

and the solution is

$$\begin{gathered}
A_0 = \pm \frac{\Gamma}{(1+k)\alpha} + e^{\mp\lambda z}\left[\frac{\pm a + \nu b}{2\alpha}\cos\omega z + \frac{-\nu a \pm b}{2\alpha}\sin\omega z\right], \\
A_1 = \frac{1}{2\alpha} e^{\mp\lambda z}[(\pm a - \nu b)\cos\omega z + (\nu a \pm b)\sin\omega z], \\
B_1 = \pm \frac{\Gamma}{(1+k)\varkappa\alpha} + e^{\mp\lambda z}\left[\frac{\pm a + \nu b}{2\varkappa\alpha}\cos\omega z + \right. \\
\left. + \frac{-\nu a + \pm b}{2\varkappa\alpha}\sin\omega z\right], \\
C_1 = \frac{\Gamma}{1+k} + e^{\mp\lambda z}(a\cos\omega z + b\sin\omega z).
\end{gathered} \tag{12}$$

Analogously, for the coefficients in series (6) with n > 1 we have

$$2nA_n + \frac{dC_n}{dz} = 0, \quad (2n-1+k)C_n + \frac{dB_n}{dz} = 0, \quad C_n = \pm \alpha A_n. \tag{13}$$

The solution of this problem, whose eigenvalues are always real, is

$$A_n = \pm \lambda a_n e^{\mp 2n\lambda z}, \quad B_n = b_n \pm \frac{2n-1+k}{2n\lambda} a_n e^{\mp 2n\lambda z}, \quad C_n = a_n e^{\mp 2n\lambda z}. \tag{14}$$

Accordingly, the functions A_n, B_n, and C_n which appear in (6) are also determined within two arbitrary constants, a_n and b_n, regardless of the value of n > 1.

We emphasize that the upper and lower signs in all these equations correspond to the minimum and maximum stress states, for which the frictional forces exerted on the medium by the walls are directed upward and downward, respectively.

The constants a and b in (10) or (12) and a_n and b_n in (14) must be found from the conditions imposed at the lower and upper boundaries of the bed. We will consider the determination of these constants separately for the various physical problems.

1. Equilibrium of a Loose Material in a Container. In this case (Fig. 1a) we are dealing with a minimum stress state; i.e., we are to use the upper signs in the equations above. Assuming that the upper boundary of the granular bed is horizontal, and assuming that it experiences a distributed normal pressure P(x) and a distributed friction force per unit area, S(x), with

$$P(x)=\sum_{n=1}^{\infty} P_n x^{2(n-1)}, \quad S(x)=\sum_{n=1}^{\infty} S_n x^{2n-1}, \tag{15}$$

we find, using the boundary conditions at the upper boundary,

$$\sigma_z = P(x), \quad \tau = S(x) \quad (z=0)$$

the following results for positive T:

$$\begin{aligned} a &= \frac{\varkappa\alpha}{\nu} P_1 - \frac{1-\nu}{2\nu} S_1 - \frac{1+\nu}{2\nu}\cdot\frac{\Gamma}{1+k}, \\ b &= -\frac{\varkappa\alpha}{\nu} P_1 + \frac{1+\nu}{2\nu} S_1 + \frac{1-\nu}{2\nu}\cdot\frac{\Gamma}{1+k}. \end{aligned} \tag{17}$$

For negative T we have

$$a = S_1 - \frac{\Gamma}{1+k}, \quad b = \frac{1}{\nu}\left(2\varkappa\alpha P_1 - S_1 - \frac{\Gamma}{1+k}\right). \tag{18}$$

In both cases the coefficients a_n and b_n from (14) are given by

$$a_n = S_n, \quad b_n = P_n - \frac{2n-1+k}{2n\lambda} S_n. \tag{19}$$

Equations (17)-(19) along with (10), (12) and (14), completely determine the solution of our problem, and this solution can easily be written explicitly.

Analysis of this solution shows that with increasing depth below the upper surface the stresses asymptotically approach limiting values independent of z and T:

$$\sigma_x \approx \frac{\Gamma}{(1+k)\alpha}, \quad \sigma_z \approx \frac{\Gamma}{(1+k)\varkappa\alpha}, \quad \tau \approx \frac{\Gamma x}{1+k} \quad \left(z \gg \frac{\alpha}{1-\nu}\right). \tag{20}$$

These values are also independent of the load applied at the upper surface of the bed and describe stresses which can arise only in a quite thick bed. It is also easy to derive approximate equations describing the stresses near the upper boundary. For example, for a bed with a free upper surface we find

$$\begin{gathered} \sigma_x \approx \frac{1-T}{2\alpha^2}\,\frac{\Gamma}{1+k}(1-x^2)z, \quad \sigma_z \approx \frac{1-T}{2\varkappa\alpha^2}\,\frac{\Gamma}{1+k} z, \\ \tau \approx \frac{1-T}{2\alpha^2}\,\frac{\Gamma}{1+k} x z^2 \quad \left(z \ll \frac{\alpha}{|1-\nu|}\right) \end{gathered} \tag{21}$$

(here we are retaining only the leading terms in the expansions in powers of z). Accordingly, both the vertical and horizontal stresses increase linearly in z with distance from the wall, while the pressure and frictional force at the walls are proportional to z^2 at small values of z; i.e., they increase much more slowly.

If $T > 0$ (i.e., if the coefficient of friction α and the quantity $\varkappa$, the measure of irreversible deformations in the system, are small), the stresses approach the limits in (20) monotonically as $z \to \infty$. On the other hand, if $T < 0$ (this case can arise if there is pronounced friction at the walls and large thrust forces due to irreversible deformations), all the stresses pass through alternating maxima and minima as functions of z (the corresponding "wavelength" is $2\pi\omega^{-1}R$). The amplitude of these maxima and minima decreases exponentially with increasing z. In this case the limiting stresses in (20) are approached in an oscillatory, nonmonotonic manner. The physical explanation is partial arch formation in the system. The various "arches" or "arcs" assume and directly transmit to the walls the forces exerted by the upper layers of the granular material; the bases of these arches approach the walls at regions where the pressure at the walls is minimal. The partial unloading of the material which occurs because of these arches leads to a relative decrease in the stresses in the lower part of the bed, explaining the minima on the z dependences of the stresses.

The nonlimiting state under discussion here can be reached in practice if there is no plastic flow at any point in the medium, i.e., if the minimum compressional stress σ_m exceeds $-\sigma_c$, where σ_c is the critical adhesion stress of the loose medium. This condition can be written as

$$2\sigma_m = \sigma_x + \sigma_z - [(\sigma_x - \sigma_z)^2 + 4\tau^2]^{1/2} > -2\sigma_c. \tag{22}$$

It is not difficult to verify that the solution found satisfies condition (22), provided that S(x) and dP(x)/dx are not too large. Otherwise, the stresses distributed over the upper boundary lead to the establishment in the upper part of the bed of a zone with a true limiting-equilibrium state, which can be analyzed by the standard methods [1, 2]. Below this zone is one in which the general relations in (10) or (12) and (14) remain valid; the shape of the boundary between these zones can be determined from the condition $\sigma_m = -\sigma_c$. The continuity conditions on the normal and tangential stresses at this boundary, which is a discontinuity in the stressed state, can be used to find the arbitrary constants in (10), (12), and (14) and thus to complete the solution of the problem in this case also. No new difficulties of a fundamental nature arise here.

Obviously, the walls cease to influence the state of the granular bed in the limit $\alpha \to 0$. In this case we find from the equations above, with P(x) = 0, S(x) = 0,

$$\sigma_x \to \varkappa\Gamma z, \quad \sigma_y \to \Gamma z, \quad \tau \to 0 \tag{23}$$

in accordance with the familiar result for a free bed.

2. Motion of a Loose Medium with a Piston. We turn now to the problem of the stresses in a loose medium supported by a piston, as shown in Fig. 1b. A detailed analysis of this problem is extremely involved and could serve as the subject of a separate paper, so here we will simply offer a brief description of the fundamental aspects of the problem. We first consider the possible states of static equilibrium of a bed with a horizontal upper boundary, at which the loading conditions in (16) are specified. At the piston surface, z = h, we have

$$\sigma_z = q(x) \quad (z = h), \quad q_1 = q(0), \quad Q = \int_0^1 q(x)(2\pi x)^k\,dx, \tag{24}$$

where q(x) is some function, and q_1 and Q are constants.

As before, the coefficients a_n and b_n from (14) are written in form (19); we determine the constants a and b from the conditions

$$\sigma_z = P_1 \quad (z = 0), \quad \sigma_z = q_1 \quad (z = h), \tag{25}$$

which follow from (16) and (23). Here we are treating q_1 as some a priori unknown parameter. Then we can write all the stresses as functions of x and z, which also depend on the parameter q_1. The requirement $C_1 = S_1$ at $z = 0$, which also follows from (16), gives us an equation from which we can determine q_1. It is easily shown that this equation has two roots, q_1^- and q_1^+, corresponding to the upper and lower signs in the equations above. The first root corresponds to the minimum stress state and was in fact calculated in Section 1. The quantity q_1^- is equal to the stress σ_z at the point $x = 0$, $z = h$, found from the solution of the problem in Section 1. The corresponding value of Q^- from (24) is equal to the force which must be applied to the piston in order to prevent the bed from moving downward.

The second root, $q_1^+ > q_1^-$, corresponds to the maximum stressed state; the critical value of the force applied to the piston, Q^+, is again determined from (24). If the force Q exerted on the piston exceeds Q^+, the bed begins to move upward; analogously, if $Q < Q-$, it begins to fall. If $Q- < Q < Q+$, the bed can be in an equilibrium state, but the problem of determining the stress field becomes statically indeterminate, for in this case there is no basis for assuming that the limiting condition for the frictional force is satisfied even at the walls. After the granular bed begins to move, the stresses in it become redistributed in such a manner that the vector sum of the gravitational force, the wall friction, and the force Q (which is now assumed given) vanishes. In this case a state of dynamic equilibrium is established. We now outline a method for solving the dynamic-equilibrium problem and analyze its results qualitatively.

There are two circumstances which fundamentally distinguish this problem from the static problem. First, the shape of the upper boundary of a moving granular bed can no longer be assumed given, e.g., horizontal, since this boundary changes during the establishment of uniform motion until it reaches an "equilibrium" shape $z = z_0(x)$. Second, the boundary friction angle turns out to depend on the velocity v: $\alpha = \alpha(v)$, where $\alpha(v)$ is some unknown function. Both the boundary shape $z_0(x)$ and the parameter v must be determined from a solution of the problem; they cannot be specified beforehand.

We assume for simplicity that the upper boundary of the bed is stress-free and that there is essentially no friction between the loose medium and the piston surface (this is the model of a "smooth" piston). Then the boundary conditions at the upper and lower surfaces of the bed are

$$\sigma_z = 0 \ (x=0,\ z=0), \quad \sigma_z = q_1 \ (x=0,\ z=h), \quad \tau = 0 \ (z=h), \tag{26}$$

where q_1 is again treated as an unknown parameter. Using the first two conditions in (26) we can find a and b precisely as we did before; the third condition yields $a_n = 0$.

The conditions that the normal and tangential stresses at the free surface vanish are written in the standard manner:

$$\sigma_x n_x + \tau n_z = 0, \quad \tau n_x + \sigma_z n_z = 0 \quad (z = z_0(x)), \tag{27}$$

where n_x and n_z are the components of the unit vector normal to this surface. The first condition in (27) leads to the nonlinear differential equation

$$\frac{dz_0}{dx} = \frac{\tau}{\sigma_x}\bigg|_{z=z_0(x)} \tag{28}$$

for the function $z_0(x)$. Noting that τ and σ_x, which appear on the right side of (28), are independent of the constants b_n, which have not yet been determined, we see that Eq. (28) can in principle always be integrated. It determines a two-parameter family of functions $z_0(x)$, which depend on both q_1 and v. It is clear from physical considerations that the only curves in this family which can be used are those corresponding to the requirement

$$|dz_0/dx| \to \infty \quad (x \to 1) \tag{29}$$

(only under this condition does the wall-friction force approach zero as the free surface of the bed is approached). From condition (29) we find a relation between the parameters:

$$q_1 = q_1(v), \tag{30}$$

which must hold in a real situation, so that we are left with only a single unknown parameter (for example, v). If $z_0(x)$ is known, the second condition in (27) permits us to find all the b_n.

Accordingly, the problem has been solved, in the sense that all the quantities of interest are expressed in terms of a single unknown parameter, v. To determine the latter we use the last condition in (24), in which Q is of course treated as a given quantity. This completes the solution of the problem.

Obviously, the states of the minimum and maximum types are achieved in the cases $Q < Q^-$ and $Q > Q^+$, respectively, so that we must use the upper or lower signs in all the equations, in accordance with the rule stated above.

In this case the condition in (22) serves as a condition that no voids appear in the medium. Analysis shows that if $Q > Q^+$ this condition holds, apparently for any parameters of the bed. If, on the other hand, we are dealing with a falling bed, then condition (22) holds only if $Q^- > Q > Q_*$, where Q_* is some critical force, which depends on the function $\alpha(v)$ and other parameters. If $Q < Q_*$ there is a discontinuity in the falling bed: The piston with the adjacent portion of the loose material falls more rapidly than does the rest of the bed. In certain particular cases, with certain values of h, $\alpha(v)$, etc., the condition $Q_* = 0$ can hold. We note that in the limit $Q \to 0$ we arrive at the limiting case of the "free" fall of a granular bed (Fig. 1c). In this case both bed surfaces are free and nonplanar.

During upward motion there is a swelling of the loose material in the upper part of the bed [the surface $z_0(x)$ is convex], while as the bed falls there is some settling (sagging) of the material (the free surface is concave).

We note in conclusion that this approach to the study of the nonlimiting states of loose media makes it possible to study not only the processes discussed above but many other important problems which are of independent interest. As an example we cite the problem of the transition of a granular bed to the fluidized state, which can be analyzed without any serious difficulty on the basis of the methods developed in the present paper, both physically and mathematically. This problem is taken up in a following paper.

NOTATION

A_j, B_j, C_j, a, b, a_j, b_j, constants; H, bed height; h, dimensionless height; k, a parameter, equal to one for the axisymmetric problem and equal to zero for the plane problem; P(x), P_j, normal load at the free surface and coefficients in its Taylor series; Q, force exerted on the piston; q(x), q_j, normal stress at the piston and the coefficients in its series expansion; R, half-width or radius of the container; S(x), S_j, tangential stresses at the free surface and the coefficients in its Taylor series; T, parameter in (8); v, velocity; x', z', horizontal and vertical coordinates; x, z, dimensionless coordinates; α, tangent of friction angle; Γ, dimensionless quantity defined in (4); γ, effective specific gravity of the medium; δ, boundary-friction angle; $\varkappa$, coefficient in the proportionality of normal stresses during homogeneous compression of the medium; λ, parameter in (11); λ_j, eigenvalues; μ_j, parameters in (9); ν, parameter in (9); σ_x, σ_z, normal stresses; σ_c, critical adhesion stress; τ, tangential stress; ω, parameter in (11).

LITERAUTRE CITED

1. V. V. Sokolovskii, Statics of a Loose Medium [in Russian], Fizmatgiz, Moscow (1960).
2. R. L. Brown and J. C. Richards, Principles of Powder Mechanics, Pergamon, New York (1966).
3. D. M. Walker, Chem. Eng. Sci., 21, 975 (1966).
4. J. K. Walters, Chem. Eng. Sci., 28, 13, 779 (1973).
5. Yu. P. Gupalo and G. P. Cherepanov, Prikl. Matim. Mekh., 31, 603 (1967).
6. Yu. P. Gupalo and G. P. Cherepanov, Izv. Akad. Nauk SSSR. Mekh. Zhidk. Gaza, No. 1, 31 (1969).
7. G. P. Cherepanov, Ind. Eng. Chem. Fund., 11, No. 9, 16 (1972).